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Forecasting crude oil price intervals and return volatility via autoregressive conditional interval models

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ABSTRACT

Crude oil prices are of vital importance for market participants and governments to make energy policies and decisions. In this paper, we apply a newly proposed autoregressive conditional interval (ACI) model to forecast crude oil prices. Compared with the existing point-based forecasting models, the interval-based ACI model can capture the dynamics of oil prices in both level and range of variation in a unified framework. Rich information contained in interval-valued observations can be simultaneously utilized, thus enhancing parameter estimation efficiency and model forecasting accuracy. In forecasting the monthly West Texas Intermediate (WTI) crude oil prices, we document that the ACI models outperform the popular point-based time series models. In particular, ACI models deliver better forecasts than univariate ARMA models and the vector error correction model (VECM). The gain of ACI models is found in out-of-sample monthly price interval forecasts as well as forecasts for point-valued highs, lows, and ranges. Compared with GARCH and conditional autoregressive range (CARR) models, ACI models are also superior in volatility (conditional variance) forecasts of oil prices. A trading strategy that makes use of the monthly high and low forecasts is further developed. This trading strategy generally yields more profitable trading returns under the ACI models than the point-based VECM.

KEYWORDS

ACI model; interval-valued crude oil prices; range; trading strategy; volatility forecast

1. Introduction

Crude oil prices are playing a significant role in the world economy. It attracts a large amount of attention from policymakers and researchers. The supply and demand for crude oil are inelastic. Therefore, crude oil prices often experience sharp and sustained fluctuations. More and more studies have found a causal link from higher crude oil prices to economic recessions, higher unemployment, inflation, and depressed consumer expenditure; see Hamilton (1983), Bernanke et al. (1997), Kilian (2009), Hamilton (2011), and Baumeister et al. (2017). In addition, stock markets are affected by crude oil prices through changes in real cash flows and expected returns. Crude oil returns and volatility have become strong predictors of those in stock markets, e.g., Chiang et al. (2015), Feng et al. (2017), and Christoffersen and Pan (2018).

A large number of econometric models have been applied to characterize the dynamics of levels and volatilities of crude oil prices; see e.g., Sadorsky (2006), Hou and Suardi (2012), and

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Efimova and Serletis (2014). Close-to-close price data are widely used, where a point-valued price observation is obtained for each time period. In fact, other information is available. For example, the highest and lowest prices during a certain period contain useful information about the level of prices. The price range is the difference between the high and low prices, and gives the width of the band within which the price fluctuates. There has been no evidence showing that point-valued closing prices contain more information on the price dynamics than the other ones.

It has been known for a long time in financial econometrics that range is a much more informative volatility proxy than a single point-valued closing price, see for instance Parkinson (1980), Chou (2005), Brownlees and Gallo (2010), Li (2016), and Liu et al. (2017). The price range plays a unique role in technical analysis. It is related to the technical indictor candlestick and stochastic oscillator. However, the focus on range alone cannot allow the recovery of the corresponding highs and lows as argued in He et al. (2010). Some trading strategies are proposed for stock and crude oil prices by modeling the joint dynamics of highs, lows and ranges to improve common technical analysis, e.g., Cheung et al. (2010), He et al. (2010), and Caporin et al. (2013). Thus, it is of interest to model both the range and its two components (i.e., the high and the low) simultaneously.

In practice, a sequence of oil prices is usually available at various time points (e.g., days) of a given time period (e.g., a month). We can thus collect the highest and lowest prices within the given time period, constructing an interval-valued time series (ITS). An interval can be characterized by two pairs of attributes: the lower (low) and upper (high) bounds, or equivalently the midpoint and range. Compared with a point-valued closing price data set, an interval data set is capable of assessing the variation as well as level information within the same time period. Rich information contained in an interval could be utilized to develop more efficient statistical inference and forecasting than a point-valued data (Han et al., 2016; Qiao et al., 2019, 2021; Sun et al., 2018, 2019, 2020a, 2020b).

Many studies in the literature have considered forecasting interval variables when both the explanatory and dependent variables are interval-valued. Univariate and bivariate methods were proposed. For a pair of attributes of interval variables (e.g., midpoint and range), modeling an interval data can be accomplished by modeling the two point-valued attributes separately with univariate models, such as exponential smoothing (Arroyo et al., 2007; Maia and de Carvalho, 2011), and univariate ARMA-X models (Brito, 2007; He and Hu, 2009; Maia et al., 2008). These univariate models utilize one attribute point-valued information of an interval data in estimating model parameters, and are not expected to be fully efficient.

To account for possible interdependence between a pair of attributes of an interval-valued time series, bivariate modeling and estimation are considered in Cheung et al. (2009), He et al. (2010), García-Ascanio and Maté (2010), Arroyo et al. (2011), González-Rivera and Lin (2013), Teles and Brito (2015), and Golan and Ullah (2017). In particular, a cointegrating relationship between highs and lows is often found, and thereby vector error correction (VEC) models of highs and lows, which are the representative of bivariate modeling in the literature, are often employed. Unlike the univariate modeling approach, VEC models can use both the high and low price information jointly to estimate model parameters. However, traditional regression models cannot guarantee that the predicted lower bound is always smaller than its upper bound. At the same time, VEC models tend to suffer from the problem of overparameterization since they allow different parameters in the two equations for a given pair of attributes of an interval.

In this paper, we apply two parsimonious autoregressive conditional interval (ACI) models to forecast monthly crude oil prices. ACI models are first proposed in Han et al. (2012) for interval-valued time series data, who also develop a minimum D_K-distance estimation method. Compared with the existing methods, ACI models and the minimum D_K-distance estimation method have several advantages. First, an ACI model captures the dynamics of an interval-valued oil price and its relationship with other interval-valued economic variables in a parsimonious interval framework. In particular, an ACI model is more parsimonious than a VEC model, and thereby the overparameterization problem can be alleviated. More accurate forecast is thus expected. Second, the minimum D_K-distance estimation method can simultaneously and efficiently utilize the midpoint, range and their correlation information to estimate model parameters, by assigning different weights on these point attributes using a kernel function, thereby enhancing model parameter efficiency. A two-stage estimation procedure is expected to produce the asymptotically most efficient estimator for ACI models among the class of minimum D_K-distance estimators. Finally, the ACI methodology can also serve as a unified framework to derive different popular point-based models of its point attributes as special cases.

Based on a series of monthly interval data over a 25-year period, we estimate two ACI models of WTI crude oil futures returns, and evaluate their out-of-sample forecasting performances with some existing popular models. Compared with Han et al. (2012), an interval-valued error correction term is first added to an ACI model, which is more parsimonious than a point-based VEC model since the former considers an interval observation as a set rather than modeling the point-valued boundaries of an interval separately. In addition, in interval-valued crude oil price forecasting, it is shown that the month-to-month changes in the midpoints at times $t-1$ and t are negatively correlated while the month-to-month changes in the range data are positively correlated. To accommodate this important empirical stylized fact, we propose a simple interval-valued data transformation.

We have the following important empirical findings:

1. The ACI models deliver better out-of-sample forecasts of monthly price intervals as well as forecasts for point-valued highs, lows, and ranges than the existing point-based time series models, including one attribute-based ARMAX models, separate time series models of two attributes, and than VEC models.
2. The ACI models are also superior in out-of-sample volatility forecasts than the threshold GARCH and range-based CARR models; the latter is a conditional autoregressive range model proposed by Chou (2005).
3. The interval-valued speculation has predictive power for oil prices and volatility in ACIX models, while this predictive ability disappears in other point-based competing methods. This highlights the gain of utilizing the valuable information contained in interval-valued data even when the interest is in range or level modeling.
4. A trading strategy that makes use of the monthly highs and lows based on ACI forecasts generally yields more profitable trading returns than the point-based VEC forecasts.

This paper is organized as follows. Section 2 describes the data and presents some basic analysis. Section 3 introduces two modified ACI models, and briefly describes the minimum D_K-distance estimation method. Section 4 introduces other competing point-based time series models in forecasting conditional mean and conditional variance of crude oil prices. Empirical results of forecasting comparison and trading strategy performance of ACI and point-valued time series models are presented in Section 5. Section 6 concludes.

2. Data description and preliminary analysis

The data we will analyze in this paper are monthly interval-valued West Texas Intermediate (WTI) crude oil futures prices $\mathbf{F}_t$ of nearby month contracts, constructed by observed daily closing prices sourced from Energy Information Administration.[1] For each month t, the lower and upper bounds of an interval-valued oil price observation $\mathbf{F}_t$ are formed using the minimum and maximum daily oil futures prices within this month. Spanning from January 1993 to March 2018,

[1]We use futures prices rather than spot prices. There are mainly two reasons. First, the futures market is a forum to disseminate crude oil information. It delivers market price signals that are essential for risk monitoring. Second, the WTI futures on NYMEX is one of the most actively traded contracts all over the world.

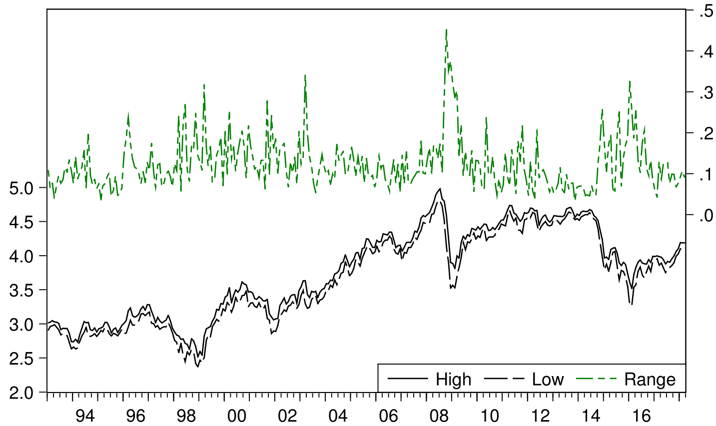

Figure 1. The monthly interval-valued WTI crude oil futures prices. *Note:* Monthly data for all series span from January 1993 to March 2018. High, Low and Range in Figure represent the logarithmic high price process (H_t), the logarithmic low price process (L_t) and the range process ($H_t - L_t$), respectively. The left axle is for High and Low, and the right axle is for Range.

the daily price data are expressed in log scale. Throughout this paper, bold letters denote interval-valued variables, while regular letters denote point-valued variables.

Figure 1 presents the plots of series of monthly logarithmic high (H_t), logarithmic low (L_t) and monthly range (R_t), where H_t and L_t form the interval-valued series of oil price $\mathbf{F}_t = [L_t, H_t]$, and $R_t = H_t - L_t$. The bounds H_t and L_t measure the trend of oil prices, while the range R_t quantifies the fluctuation of oil prices. Figure 1 shows that the price fluctuations are dramatic in the crude oil market, and in particular, ranges tend to spike upward while the oil prices fall down, indicating a negative correlation between the trend and volatility contained in the interval oil price data. Furthermore, it is shown that the interval-valued oil prices in different time periods may have the same range but distinct price levels, while the similar price levels occurring in different periods may be accompanied with totally different ranges, suggesting that the interval-valued observations contain additional important information to the point-valued observations.

A summary of statistical characteristics of the midpoint ($M_t = (L_t + H_t)/2$), range (R_t), the first difference (ΔH_t) of H_t, the first difference (ΔL_t) of L_t and the first difference (ΔFA_t) of logarithmic monthly average futures price is provided in Table 1.[2] We observe that the intra-month oil price fluctuation measured by the sample mean (0.1233) of monthly ranges is about 31 times higher than the month-to-month change in highs (or lows) measured by the sample mean (0.0039) of ΔH_t (or ΔL_t), and is 30 times higher than the month-to-month change in monthly average price level measured by the sample mean (0.0040) of ΔFA_t. On the other hand, the dispersion of ranges is a little smaller than the changes in highs, lows and average prices, respectively.

Augmented Dickey–Fuller test results find that H_t and L_t are nonstationary in levels but stationary in first differences.[3] Previous studies find a cointegrating relationship between H_t and L_t, e.g., in He et al. (2010) and Cheung et al. (2009). If a cointegrating relationship exists between H_t and L_t, which will be tested in this paper, we could use the VEC model as a benchmark. Meanwhile, an error correction term could be added to the ACI model to capture the short-term dynamics of crude oil returns. Table 2 presents the results of Johansen's trace and maximal eigenvalue tests. Cointegration of rank 1 is indeed present between H_t and L_t, implying that the lagged error correction term EC_{t-1} contains valuable information for future movement of crude oil prices.

[2]Monthly average futures price is the average daily futures prices over the trading days within each month.
[3]The results are not reported here for space but are available upon request.

Table 1. Some basic statistics.

	M_t	R_t	ΔL_t	ΔH_t	ΔFA_t
Mean	3.7102	0.1233	0.0039	0.0039	0.0040
Std.	0.6480	0.0637	0.0934	0.0811	0.0821
Minimum	2.4441	0.0335	−0.3819	−0.3585	−0.3121
Maximum	4.8922	0.4515	0.2719	0.2814	0.2149

Note: Monthly data for all series span from January 1993 to March 2018. M_t,R_t,L_t,H_t are the midpoint, the range, the highest, and the lowest of crude oil log futures prices within a month, respectively. FA_t is the natural logarithm of the monthly average crude oil futures price. Std. is the monthly standard deviation.

Table 2. Cointegration test results.

Hypothesis	EIGEN	TRACE	H_t	L_t	Lag
$r = 0$	33.26***	35.76***			1
$r \leq 1$	2.51	2.51			1
Cointegration parameters			1	0.8897	

Note: The maximal eigenvalue and trace statistics are presented in the columns denoted by 'EIGEN' and 'TRACE' respectively. 'Lag' is the lag length in the VECM. There is no deterministic trend in the data. Asterisks ***, **, and * indicate significance at the 1%, 5% and 10% significance levels respectively.

In order to forecast future crude oil prices, we include an explanatory variable called speculation $\mathbf{SPE}_t$, which is formed by using weekly data of long positions held by non-commercials of crude oil in the futures market of NYMEX. The long positions of non-commercials is used in De Roon et al. (2000) to measure the hedging pressure, which the futures risk premia depend on. The point-valued speculation, either at monthly or weekly level, has been shown to be significant in explaining the dynamics of crude oil prices in Wang (2003), Sanders et al. (2004), and Merino and Ortiz (2005). It is thus expected that interval-valued speculation is also useful for forecasting oil price intervals, as confirmed in our empirical results below. The data are sourced from U.S. Commodity Futures Trading Commission (CFTC). Minimum and maximum weekly positions form a monthly interval-valued speculation observation in a given month. A number of other variables, like surplus production capacity and OECD petroleum inventory, may be also helpful to forecast oil prices, but we do not consider them in this paper.

3. ACI models of crude oil prices

3.1. Model specification

First proposed in Han et al. (2012), the autoregressive conditional interval models with orders (p, q) (ACI(p, q) thereof) is an interval version of ARMA models to capture the conditional mean dynamics of a univariate ITS. The class of ACI models has been further extended to model an interval-valued vector process in Han et al. (2016), and to examine the possible nonlinear features of a univariate interval process in Sun et al. (2018). Given that Section 2 shows that the error correction term EC_{t-1} is useful to explain dynamic movements of both highs and lows, we propose an ACI(2, 2) model of the differenced interval-valued crude oil prices $\mathbf{F}_t = [L_t, H_t]$ that includes interval-valued variable $\mathbf{EC}_t$[4]:

$$\Delta \mathbf{F}_t = \alpha_0 + \beta_0 I_0 + \gamma_1 \mathbf{EC}_{t-1} + \sum_{j=1}^{2} \beta_j \Delta \mathbf{F}_{t-j} + \sum_{j=1}^{2} \theta_j \Delta \mathbf{u}_{t-j} + \mathbf{u}_t, \tag{3.1}$$

[4]To determine the lag orders in an ACI model for oil prices, we can refer to the implied point-valued equation (3.8). An ARMA(2,2) is adequate for both midpoint and range. Similarly, second-lagged values of the differenced speculation variable are significant to predict the crude oil midpoint.

and with the inclusion of the exogenous interval-valued speculation variable $\mathbf{SPE}_t$:

$$\Delta\mathbf{F}_t = \alpha_0 + \beta_0 I_0 + \gamma_1 \mathbf{EC}_{t-1} + \sum_{j=1}^{2} \beta_j \Delta\mathbf{F}_{t-j} + \sum_{j=1}^{2} \theta_j \Delta\mathbf{u}_{t-j} + \gamma_2 \Delta\mathbf{SPE}_{t-2} + \mathbf{u}_t, \tag{3.2}$$

where $\mathbf{F}_t, \mathbf{EC}_{t-1}$, and $\mathbf{SPE}_t$ are interval-valued processes; Δ denotes the Hukuhara difference of an interval-valued process, e.g.,

$$\Delta\mathbf{F}_t = [L_t - L_{t-1}, H_t - H_{t-1}] = [\Delta L_t, \Delta H_t], \tag{3.3}$$

which is a stationary interval-valued process though $\mathbf{F}_t$ is not; α_0, β_j $(j = 0, 1, 2), \gamma_j$ and θ_j $(j = 1, 2)$ are scalar-valued unknown parameters, and $I_0 = [-\frac{1}{2}, \frac{1}{2}]$ is a constant unit interval; $\mathbf{u}_t = [u_{L,t}, u_{H,t}]$ is an interval martingale difference sequence with respect to the information set I_{t-1} at time $t-1$ such that $E(\mathbf{u}_t|I_{t-1}) = [0, 0]$; $\alpha_0 + \beta_0 I_0 = [\alpha_0 - \frac{1}{2}\beta_0, \alpha_0 + \frac{1}{2}\beta_0]$ is an interval intercept; $\mathbf{EC}_{t-1} = [\frac{1}{2}EC_{t-1}, \frac{3}{2}EC_{t-1}]$[5] is the lagged interval-valued error correction term, where EC_{t-1} is the lagged error correction term induced from the long-run equation between H_t and L_t; $\mathbf{SPE}_t$ is the interval-valued speculation introduced in Section 2.

A few remarks need to be made here. First of all, an ACI model serves direct interest to forecast crude oil price/return intervals during a given time period, based on the past history of crude oil price interval data. In particular, each interval observation in our model is an inseparable set of ordered numbers that includes not only the naturally ordered intervals (i.e., the left bound does not exceed the right bound), but also the reversely ordered intervals (i.e., the left bound exceeds the right bound); see details in Han et al. (2012, 2016) and Sun et al. (2018). This extension is suitable in modeling interval-valued crude oil process, because one could follow the popular log-difference transformation of a price interval time series to obtain an interval version of stationary log return interval process as in (3.3), where return intervals $\Delta\mathbf{F}_t = [\Delta L_t, \Delta H_t]$ with reverse order are frequently observed.

Next, in addition to forecast intervals, an ACI model is also useful in forecasting range-based volatility of crude oil prices. By taking the difference between interval boundaries in model (3.2), it yields an ARMAX type range model, namely:

$$\Delta R_t = \beta_0 + \gamma_1 EC_{t-1} + \sum_{j=1}^{2} \beta_j \Delta R_{t-j} + \sum_{j=1}^{2} \theta_j u_{R,t-j} + \gamma_2 R_{\Delta\mathbf{SPE},t-2} + u_{R,t}, \tag{3.4}$$

where R_t and $R_{\Delta\mathbf{SPE},t-2}$ are the ranges of the original interval processes respectively; $u_{R,t}$ is the range of $\mathbf{u}_t$ satisfies $E(u_{R,t}|I_{t-1}) = 0$. The model (3.4) can be used to forecast the range of a time series, where parameter estimation can be obtained by using conventional point-based time series techniques, e.g., conditional least squares (CLS). Note that the intercept parameter α_0 of the ACI model in (3.2) cannot be identified in the range model (3.4), since the point-based CLS estimator utilizes the range sample only. Figure 1 suggests that the range and level of an interval-valued oil price are correlated, thereby more efficient estimation and so better range forecasts using an ACI model than the range model (3.4) could be obtained, as is confirmed in our empirical studies. This is one advantage of the ACI model in (3.2) over the range model in (3.4) even if the interest is in range forecasting.

Third, Yang et al. (2016) have shown that the speculation index, as a proxy of crude oil market liquidity, is significant in capturing the dynamics of crude oil prices. Hence, we include the lagged interval speculation term in the ACI model (3.2) to evaluate the predictability of speculation by comparing the forecasting accuracy of the ACI models in (3.1) and (3.2) in the subsequent sections.

[5]$\mathbf{EC}_{t-1} = [\frac{1}{2}EC_{t-1}, \frac{3}{2}EC_{t-1}]$ ensures that EC_{t-1} has the same coefficient in the range equation (3.4) and the midpoint equation (3.5).

Like the derived range model in (3.4), a midpoint model can also be derived from the original ACI model in (3.2):

$$\Delta M_t = \alpha_0 + \gamma_1 EC_{t-1} + \sum_{j=1}^{2} \beta_j \Delta M_{t-j} + \sum_{j=1}^{2} \theta_j u_{M,t-j} + \gamma_2 M_{\Delta \mathbf{SPE},t-2} + u_{M,t}, \tag{3.5}$$

where M_t and $M_{\Delta \mathbf{SPE},t-2}$ are the midpoints of the original interval processes respectively; $u_{M,t}$ is the midpoint of $\mathbf{u}_t$ satisfies $E(u_{M,t}|I_{t-1}) = 0$. Note that the scale parameter β_0 cannot be identified in the midpoint model. Here, a practical issue might arise concerning our sample data. Since the true data generating process (DGP) of the observed crude oil prices is unknown, parameter estimates in the derived range model in (3.4) and the derived midpoint model in (3.5) may have opposite signs if these two models are estimated separately. Indeed, data inspection of our sample shows that for the midpoint data, ΔM_{t-1} and ΔM_t are negatively correlated, while for range data ΔR_{t-1} and ΔR_t are positively correlated. Therefore, assuming that ΔM_{t-1} and ΔR_{t-1} share a same autoregressive parameter β_1 as in the ACI model (3.2) is no longer tenable. To address this issue, we propose a simple data transformation of $\Delta \mathbf{F}_{t-1}$, which substitutes the interval-valued oil price $\mathbf{F}_{t-1} = [L_{t-1}, H_{t-1}]$ with a new interval-valued variable $\mathbf{F}^*_{t-1} = [-H_{t-1}, -L_{t-1}]$. Intuitively, $\mathbf{F}^*_{t-1}$ can be viewed as a negative of $\mathbf{F}_{t-1}$. As a result, the range of $\Delta \mathbf{F}^*_{t-1}$ is ΔR_{t-1} and the midpoint of $\Delta \mathbf{F}^*_{t-1}$ is $-\Delta M_{t-1}$. Thus, the ACI model in (3.2) can be modified as follows[6]

$$\Delta \mathbf{F}_t = \alpha_0 + \beta_0 I_0 + \gamma_1 \mathbf{EC}_{t-1} + \beta_1 \Delta \mathbf{F}^*_{t-1} + \beta_2 \Delta \mathbf{F}_{t-2} + \sum_{j=1}^{2} \theta_j \Delta \mathbf{u}_{t-j} + \gamma_2 \Delta \mathbf{SPE}_{t-2} + \mathbf{u}_t. \tag{3.6}$$

The coefficients of $\Delta \mathbf{F}^*_{t-1}$ in the derived midpoint and range models now share the same sign, which makes the modified ACI in model (3.6) more sensible. Sometimes economic theories can help check signs of some parameters. Otherwise, if relevant economic theories do not exist, a plausible approach to checking the correlation between the regressand and regressor is running CLS estimation for the midpoint and range models separately, using the original interval sample data. The CLS estimators are consistent but not expected to be most efficient, because they make use of the range and midpoint information separately.

Interestingly, the ACI models (e.g., model (3.6)) can be used to derive some important point-based time series models as special cases. For example, one can immediately derive a bivariate point-valued time series model for the lower and upper bounds:

$$\begin{cases} \Delta L_t = c_L + \gamma_1 L_{EC,t-1} - \beta_1 \Delta H_{t-1} + \beta_2 \Delta L_{t-2} + \sum_{j=1}^{2} \theta_j u_{L,t-j} + \gamma_2 L_{\Delta \mathbf{SPE},t-2} + u_{L,t}, \\ \Delta H_t = c_H + \gamma_1 H_{EC,t-1} - \beta_1 \Delta L_{t-1} + \beta_2 \Delta H_{t-2} + \sum_{j=1}^{2} \theta_j u_{H,t-j} + \gamma_2 H_{\Delta \mathbf{SPE},t-2} + u_{H,t}, \end{cases} \tag{3.7}$$

where $c_L = \alpha_0 - \frac{1}{2}\beta_0, c_H = \alpha_0 + \frac{1}{2}\beta_0, L_{EC,t-1} = \frac{1}{2} EC_{t-1}, H_{EC,t-1} = \frac{3}{2} EC_{t-1}, L_{\Delta \mathbf{SPE},t}$ and $H_{\Delta \mathbf{SPE},t}$ are the lower and upper bounds of $\Delta \mathbf{SPE}_t$, and $E(u_{L,t}|I_{t-1}) = E(u_{H,t}|I_{t-1}) = 0$. Note that there are constraints on identical parameters except for the intercepts.

[6]The modified ACI model can be extended to a more general ACI(p, q) model of order (p, q):

$$\Delta \mathbf{F}_t = \alpha_0 + \beta_0 I_0 + \gamma_1 \mathbf{EC}_{t-1} + \sum_{j=1}^{p} \beta_j \Delta \mathbf{F}_{t-j} + \sum_{j=1}^{p} \beta^*_j \Delta \mathbf{F}^*_{t-j} + \sum_{j=1}^{q} \gamma_j \mathbf{u}_{t-j} + \sum_{j=1}^{q} \gamma^*_j \mathbf{u}^*_{t-j} + \mathbf{u}_t,$$

where $\mathbf{u}^*_{t-j} = [-u_{H,t-j}, -u_{L,t-j}]$. Among other things, this can be used to address the important empirical stylized facts in interval data, such as the fact that the midpoints at times $t - j$ and t are negatively correlated while the ranges are positively correlated.

Analogous to the model (3.7), a bivariate point-valued time series model for the midpoint and range processes from the ACI model in (3.6) can also be obtained:

$$\begin{cases} \Delta M_t = \alpha_0 + \gamma_1 EC_{t-1} - \beta_1 \Delta M_{t-1} + \beta_2 \Delta M_{t-2} + \sum_{j=1}^{2} \theta_j u_{M,t-j} + \gamma_2 M_{\Delta \mathbf{SPE},t-2} + u_{M,t}, \\ \Delta R_t = \beta_0 + \gamma_1 EC_{t-1} + \beta_1 \Delta R_{t-1} + \beta_2 \Delta R_{t-2} + \sum_{j=1}^{2} \theta_j u_{R,t-j} + \gamma_2 R_{\Delta \mathbf{SPE},t-2} + u_{R,t}, \end{cases} \tag{3.8}$$

Different from separately modeling the midpoints and ranges respectively, this is a bivariate modeling for the two-point processes with constraints on identical parameters except for the intercepts, estimating all parameters simultaneously. It is observed that ACI models can be used to capture some well-known empirical stylized facts, including mean-reversion. For example, $\beta_1 > 0$ indicates the mean-reversion that historical returns eventually will revert to the long-run mean.

These derived point-valued models of endpoints in (3.7) and (3.8) have a bivariate modeling flavor, but they cannot capture the most crucial feature of our interval modeling approach in (3.6), namely, we treat an interval as a set of ordered numbers. We can follow the spirit of Han et al. (2012) to estimate parameters for the parsimonious modified ACI model (3.6) by minimizing the distance between the modified ACI model and interval data. Interestingly, this distance measure between intervals is based on the random sets theory, which employs not only the information on distances between boundaries of an ITS but also the information on distances between interior points. As a result, this interval modeling framework has an informational advantage of utilizing rich information contained in interval data over a bivariate point-valued model. This is expected to enhance model forecasting accuracy.

3.2. Estimation

Parameter estimation of the modified ACI model (3.6) can be accomplished by the minimum D_K-distance estimation method developed in Han et al. (2012). Based on the D_K-distance measure for sets, one can estimate unknown parameters $\phi = (\alpha_0, \beta_0, \beta_1, \beta_2, \theta_1, \theta_2, \gamma_1, \gamma_2)'$ in model (3.6) by minimizing the squared D_K distance between the interval model and the observed interval data:

$$\hat{Q}_T(\phi) = \sum_{t=1}^{T} D_K^2 \left[\Delta \mathbf{F_t}, \Delta \hat{\mathbf{F}}_t(\phi) \right], \tag{3.9}$$

where $\Delta \hat{\mathbf{F}}_t(\phi)$ is the fitted value of $\Delta \mathbf{F}_t(\phi)$ based on model (3.6), and $D_K^2(A, B)$ denotes the square of the D_K distance between intervals $A = [A_l, A_u]$ and $B = [B_l, B_u]$:

$$D_K^2(A, B) = d' \mathbf{K} d \tag{3.10}$$

where $d = (A_u - B_u, -(A_l - B_l))'$, $\mathbf{K}$ is a 2×2 matrix with $\mathbf{K}_{11} = K(1, 1), \mathbf{K}_{22} = K(-1, -1)$, $\mathbf{K}_{12} = \mathbf{K}_{21} = K(1, -1) = K(-1, 1)$. We assume that $K(u, v)$ is a symmetric positive definite weighting function such that for $u, v \in \mathbf{S}^0 = \{u \in \mathbf{R}^1, |u| = 1\} = \{1, -1\}$,

$$\begin{cases} K(1, 1) > 0, K(1, -1) = K(-1, 1) \\ K(1, 1) K(-1, -1) > K(1, -1)^2. \end{cases} \tag{3.11}$$

The objective function has a simple quadratic form as in (3.10), which can be numerically calculated given the boundaries of intervals. Han et al. (2012) have shown that the D_K metric considers the set of the absolute differences between all possible pairs of points (extreme and interior points) in intervals A and B, with a proper weighting function implied by $K(u, v)$. Thus, it utilizes interval information more efficiently than conventional point-valued time series techniques, e.g., model (3.4) with CLS.

Different choices of kernel K will deliver different estimators for ϕ, and all of them are consistent and asymptotically normal, provided the kernels satisfy (3.11). Following Han et al. (2012), we employ a feasible two-stage minimum D_K-distance estimator that is asymptotically most efficient among the class of minimum D_K distance estimators. First, we obtain a preliminary consistent estimator $\hat{\phi}$. For example, it can be a minimum D_K-distance estimator with an arbitrary prespecified kernel K satisfying (3.11). Then, we compute the estimated residuals $\{\hat{\mathbf{u}}_t(\hat{\phi})\}$ and construct an optimal kernel estimator $\hat{K}^{opt}(1,\ 1)=T^{-1}\sum_{t=1}^{T}\hat{u}_{L,t}^2(\hat{\phi}), \hat{K}^{opt}(1,-1)=\hat{K}^{opt}(-1,\ 1)= T^{-1}\sum_{t=1}^{T}\hat{u}_{L,t}(\hat{\phi})\hat{u}_{H,t}(\hat{\phi})$, and $\hat{K}^{opt}(-1,\ -1)=T^{-1}\sum_{t=1}^{T}\hat{u}_{H,t}^2(\hat{\phi})$. Next, we obtain a second-stage minimum D_K-distance estimator with the choice of $K=\hat{K}^{opt}$:

$$\hat{\phi}^{opt} = \arg\min_{\phi\in\Phi} T^{-1}\sum_{t=1}^{T} D^2_{\hat{K}^{opt}}\left[\Delta\mathbf{F_t}, \Delta\hat{\mathbf{F}}_t(\phi)\right].$$

Han et al. (2012) show that when K^{opt} is used, the objective function of the minimum D_K-distance estimator becomes

$$var(u_{L,t})\sum_{t=1}^{T}\hat{u}_{H,t}^2(\phi) + \text{var}\ (u_{H,t})\sum_{t=1}^{T}\hat{u}_{L,t}^2(\phi) - 2\ \text{cov}\ (u_{L,t}, u_{H,t})\sum_{t=1}^{T}\hat{u}_{L,t}(\phi)\hat{u}_{H,t}(\phi).$$

Thus, K^{opt} downweights the sample squared distance components that have larger sampling variations. Specifically, it discounts the sum of squared residuals of the upper (lower) bound if the upper (lower) bound disturbance $u_{H,t}$ $(u_{L,t})$ has a large variance. The use of K^{opt} also corrects correlation between the left and right bound disturbances. Such heteroskedasticity and correlation corrections are similar in spirit to the optimal weighting matrix in GLS, and improve estimation efficiency.

3.3. Special cases of ACI models and interval-based estimation

Combining the midpoint and range time series models (e.g., (3.4) and (3.5)) yields an alternative approach to forecasting intervals. In fact, one can estimate the univariate ARMAX type models using CLS based on midpoint and range data separately, and then construct a one-step ahead interval forecast as $[\Delta\hat{M}_t - \frac{1}{2}\Delta\hat{R}_t, \Delta\hat{M}_t + \frac{1}{2}\Delta\hat{R}_t]$, where $\Delta\hat{M}_t$ and $\Delta\hat{R}_t$ are one-step-ahead point predictors for ΔM_t and ΔR_t, respectively.

CLS estimators are convenient and they can consistently estimate most parameters in the ACI model. However, in addition to the failure in identifying level parameter α_0 or scale parameter β_0, these estimators are not most efficient because they make use of the range and level sample information separately. Han et al. (2012) show that the CLS estimators using one attribute of an interval time series are special cases of the minimum D_K distance estimator, with specific choices of kernel $K(u, v)$. When the kernel K_M with $-K_M(1, -1) = K_M(1,1) = K_M(-1, -1)$ is used, the minimum D_K-distance estimator solves:

$$\hat{\phi}_M = \arg\min_{\phi_M\in\Phi}\sum_{t=1}^{T} D^2_{K_M}(\Delta\mathbf{F_t}, \Delta\hat{\mathbf{F}}_\mathbf{t}) = \arg\min_{\phi_M\in\Phi}\sum_{t=1}^{T}\left[\Delta M_t - \Delta\hat{M}_t(\phi)\right]^2, \tag{3.12}$$

which boils down to the CLS estimator of the midpoint time series model (3.5). Similarly, when the kernel K_R with $K_R(1, -1) = K_R(1,1) = K_R(-1, -1)$ is used, the minimum D_K-distance estimator becomes the CLS estimator for the range time series model:

$$\hat{\phi}_R = \arg\min_{\phi_R \in \Phi} \sum_{t=1}^{T} D^2_{K_R}(\Delta \mathbf{F_t}, \Delta \hat{\mathbf{F}}_\mathbf{t}) = \arg\min_{\phi_R \in \Phi} \sum_{t=1}^{T} \left[\Delta R_t - \Delta \hat{R}_t(\phi)\right]^2. \tag{3.13}$$

With the efficiency gain in estimation, it is expected that using the ACI methodology with the two-stage minimum D_K distance estimator outperforms the univariate time series models for midpoints and ranges, in terms of the forecast accuracy. This is confirmed in our empirical study on crude oil prices.

To estimate all parameters of the modified ACI model (3.6), one can consider the bivariate point-based model (3.7) for the lower and upper bounds derived from the ACI model in (3.6). When the kernel K_{LH} with $K_{LH}(1,1) = K_{LH}(-1,-1) > 0, K_{LH}(1,-1) = 0$ is used, the minimum D_K-distance estimator minimizes the combined sums of squared residuals for the lower and upper bound models together:

$$\hat{\phi}_{LH} = \arg\min_{\phi_{LH} \in \Phi} \sum_{t=1}^{T} D^2_{K_{LH}}(\Delta \mathbf{F_t}, \Delta \hat{\mathbf{F}}_\mathbf{t}), \tag{3.14}$$

where $D^2_{K_{LH}}(\Delta \mathbf{F_t}, \Delta \hat{\mathbf{F}}_\mathbf{t}) = \left[\Delta L_t - \Delta \hat{L}_t(\phi)\right]^2 + \left[\Delta H_t - \Delta \hat{H}_t(\phi)\right]^2$. This is the constrained conditional least square (CCLS) estimator for the lower and upper bound models. By using the bivariate sample data for the two boundaries and the kernel K_{LH}, the CCLS estimator assigns the same importance on the information of the two boundaries and ignores the possible correlation between them in the objective function. Thus, the CCLS estimator is not expected to be the most efficient estimator for an ACI model. Furthermore, we can also construct a one-step-ahead interval predictor $[\Delta \hat{L}_t, \Delta \hat{H}_t]$ given the past information set I_{t-1}, based on (3.7). They are less accurate than the interval forecasts produced from the modified ACI model in (3.6) with the two-stage minimum D_K distance estimator, as confirmed in our empirical study.

Similarly, we can consider the bivariate time series model (3.8) for the midpoint and range processes derived from the ACI model in (3.6). When the kernel K_{MR} with $K_{MR}(1,1) = K_{MR}(-1,-1) = 5$, and $K_{MR}(1,-1) = 3$ is used, the minimum D_K-distance estimator becomes the CCLS estimator that minimize the total sums of squared residuals for the midpoint and range models together:

$$\hat{\phi}_{MR} = \arg\min_{\phi_{MR} \in \Phi} \sum_{t=1}^{T} D^2_{K_{MR}}(\Delta \mathbf{F_t}, \Delta \hat{\mathbf{F}}_\mathbf{t}), \tag{3.15}$$

where $D^2_{K_{MR}}(\Delta \mathbf{F_t}, \Delta \hat{\mathbf{F}}_\mathbf{t}) = \left[\Delta M_t - \Delta \hat{M}_t(\phi)\right]^2 + \left[\Delta R_t - \Delta \hat{R}_t(\phi)\right]^2$. Note that K_{MR} assigns the same importance on the midpoints and ranges but ignores the possible correlation between them in the objective function. We can construct the one-step-ahead interval predictor $[\Delta \hat{M}_t - \frac{1}{2}\Delta \hat{R}_t, \Delta \hat{M}_t + \frac{1}{2}\Delta \hat{R}_t]$ given the CCLS estimator $\hat{\phi}_{MR}$ and model (3.8). However, these forecasts are expected to be less accurate than the interval forecasts produced from the modified ACI model in (3.6) with the two-stage minimum D_K distance estimator, as confirmed in our empirical study.

4. Point-based models of crude oil prices

4.1. Point-based forecasting models for lows and highs of crude oil prices

This section presents two benchmark models as point-based counterparts of the interval-based ACI models in forecasting the dynamics of crude oil prices.

ARMAX models for midpoint and range. As mentioned in Section 3, forecasting the interval-valued crude oil prices can be accomplished by predicting two point-valued attributes, i.e., midpoints and ranges, separately using the ARMAX models in (3.5) and (3.4), respectively. Here we re-write the benchmark models to explicitly allow the possibility that the parameters are not identical in the midpoint and the range models if they are not considered as being derived from an ACI model:

$$\begin{cases} \Delta M_t = \alpha_0 + \gamma_{m1} EC_{t-1} + \sum_{j=1}^{2} \beta_{mj} \Delta M_{t-j} + \sum_{j=1}^{2} \theta_{mj} u_{M,t-j} + u_{M,t}, \\ \Delta R_t = \beta_0 + \gamma_{r1} EC_{t-1} + \sum_{j=1}^{2} \beta_{rj} \Delta R_{t-j} + \sum_{j=1}^{2} \theta_{rj} u_{R,t-j} + u_{R,t}, \end{cases} \tag{4.1}$$

and by including the speculation factor we obtain:

$$\begin{cases} \Delta M_t = \alpha_0 + \gamma_{m1} EC_{t-1} + \sum_{j=1}^{2} \beta_{mj} \Delta M_{t-j} + \sum_{j=1}^{2} \theta_{mj} u_{M,t-j} + \gamma_{m2} \Delta M_{\mathbf{SPE},t-2} + u_{M,t}, \\ \Delta R_t = \beta_0 + \gamma_{r1} EC_{t-1} + \sum_{j=1}^{2} \beta_{rj} \Delta R_{t-j} + \sum_{j=1}^{2} \theta_{rj} u_{R,t-j} + \gamma_{r2} \Delta R_{\mathbf{SPE},t-2} + u_{R,t}, \end{cases} \tag{4.2}$$

Different coefficients for midpoint and range processes can be estimated by the CLS method using the midpoint and range data separately, as the special cases of the minimum D_K-distance estimator discussed in Section 3.3. Compared with the ACI model, the benchmark models in (4.1) and (4.1) do not simultaneously utilize rich information of midpoint and range contained in the interval sample when estimating model parameters.

Vector error correction model. The vector error correction (VEC) model is an alternative to forecast two attributes of interval time series, i.e., highs and lows, using a bivariate time series model that can explore correlations between them when estimating model parameters. It also allows different coefficients in the equations of the two attributes. As a result, it is less parsimonious than an ACI model. The VEC model is written as:

$$\begin{cases} \Delta L_t = \alpha_{10} + \alpha_{L1} EC_{t-1} + \sum_{j=1}^{2} \beta_{L1j} \Delta L_{t-j} + \sum_{j=1}^{2} \beta_{L2j} \Delta H_{t-j} + u_{L,t}, \\ \Delta H_t = \alpha_{20} + \alpha_{H1} EC_{t-1} + \sum_{j=1}^{2} \beta_{H1j} \Delta L_{t-j} + \sum_{j=1}^{2} \beta_{H2j} \Delta H_{t-j} + u_{H,t}, \end{cases} \tag{4.3}$$

where (H_t, L_t) is the bivariate vector of the logarithms of high and low within month t, and (u_{Lt}, u_{Ht}) is the bivariate disturbance. VEC models have been used to forecast stock prices or crude oil prices in Cheung et al. (2009), He et al. (2010), and Arroyo et al. (2011). Similarly, monthly logarithmic highs and lows of speculation could be included in the VEC model.

4.2. Point-based forecasting models for ranges of crude oil prices

As a quantitative measure of risk, volatility modeling is important to understand the nature of the dynamics of volatilities. It is closely related to the stability of commodity markets, financial markets and the real economy. In order to evaluate the performance of the ACI models in forecasting the monthly conditional variance of crude oil prices, we consider Glosten et al.'s (1993) threshold GARCH model and Chou's (2005) conditional autoregressive range (CARR) model as benchmarks.

Threshold GARCH model. It is well known that crude oil returns have some stylized facts, such as fat tails and leverage effects. As proposed by Glosten et al. (1993), the GJR model has several advantages over other GARCH models, particularly in capturing asymmetric features (e.g., leverage effect) in volatility. Therefore, it has been frequently used in the literature (Wen et al., 2012;

Du and He, 2015; Wang et al., 2016). We consider a point-valued GJR(1,1) model of the first difference of monthly average oil price ΔFA_t :

$$\Delta FA_t = c + \alpha \Delta FA_{t-1} + \varepsilon_t, \tag{4.4}$$

$$\varepsilon_t = \sqrt{h_t} z_t, \tag{4.5}$$

$$h_t = \delta_0 + \delta_1 \varepsilon_{t-1}^2 + \gamma \varepsilon_{t-1}^2 d_{t-1} + \delta_2 h_{t-1}, \tag{4.6}$$

$$z_t \sim skewed - t(\eta, \phi), \tag{4.7}$$

where the indicator $d_{t-1} = 1$ when $\varepsilon_{t-1} < 0$, denoting bad news; otherwise $d_{t-1} = 0$ indicating good news. We can also include the lagged monthly speculation variables in the threshold GARCH model and obtain a threshold GARCHX model. (4.4) decomposes ΔFA_t into a conditional mean and an innovation ε_t, (4.5) defines the standardized error z_t, and (4.6) describes the dynamics of the conditional variance h_t of ΔFA_t. The coefficient γ measures the difference between the asymmetry effects of good news and bad news on the conditional variance. In (4.7), the standardized error z_t is assumed to follow the skewed-t distribution. This distribution has two parameters: the degree of freedom parameter η which controls the tail thickness and the skewness parameter ϕ which controls the degree of asymmetry. If $\phi > 0$, the variable is skewed to the right, and vice-versa when $\phi < 0$. When $\phi = 0$, the standardized Student's t distribution is obtained. When $\eta \to \infty$ and $\phi = 0$, we obtain the standard normal distribution. The skewed t distribution allows for a rich set of properties of financial returns, therefore, it has been frequently used as a distribution model for z_t in the literature (e.g., Patton, 2013).

The range is a much more efficient volatility proxy, which is known at least since the works of Parkinson (1980) and Chou (2005). Andersen and Bollerslev (1998) show that daily range contains approximately the same informational content as sampling intra-daily returns every four hours. In fact, both the logarithmic range and the logarithmic absolute return are linear logarithmic volatility proxies. However, the standard deviation of the logarithmic range is about one quarter of the standard deviation of the logarithmic absolute return; see e.g. Brandt and Jones (2006). Therefore, the range is a much more informative volatility proxy. Range has provided better out-of-sample forecasts of volatility than a standard GARCH model, as shown in Chou (2005). Therefore, we also use CARR model proposed by Chou (2005) as another benchmark.

CARR Model. CARR is a range-based volatility model, which is described as:

$$R_t = \lambda_t \varepsilon_t, \tag{4.8}$$

$$\lambda_t = \varpi + \sum_{i=1}^{q} \alpha_i R_{t-i} + \sum_{j=1}^{p} \beta_j \lambda_{t-j}, \tag{4.9}$$

$$\varepsilon_t | I_{t-1} \sim f(\cdot), \tag{4.10}$$

where R_t is the range at time t, λ_t is the conditional mean of R_t, and ε_t is a standardized nonnegative disturbance. To ensure positivity of λ_t, the coefficients in the conditional mean equation (4.9) are assumed to be positive. The normalized range ε_t follows a density function $f(\cdot)$ with a unit mean. A natural choice for the distribution of ε_t is the standard exponential distribution. Parameters of the CARR model can be consistently estimated by the quasi-maximum likelihood method.

5. Empirical results

5.1. Forecasting crude oil prices

In this section, we assess the one-step-ahead forecast performances of interval-based ACI models and various popular point-based time series models introduced in Section 4.1 in forecasting various attributes of oil prices including monthly price intervals and the midpoints, ranges, highs, and lows of crude oil prices.

A rolling estimation scheme is adopted for a 10 year (120 months) rolling window with the first estimation sample spanning from January 1993 to December 2002. We conduct the one-step-ahead out-of-sample forecasts from January 2003 to March 2018 and from January 2008 to March 2018, respectively. There are $S = 183$ and $S = 123$ forecasting periods for each model, respectively. For the fixed rolling window (120 months), we compare the following models:

ARMAX: Univariate ARMAX models for midpoint and range processes separately, with CLS estimation;
Bivariate MR: Bivariate models for the midpoints and ranges, with CCLS estimation;
Bivariate LH: Bivariate models for lower and upper bounds, with CCLS estimation;
VEC: VEC models for monthly highs and lows;
ACI (one-stage): ACI model with an arbitrary choice of kernel K, where $K(1,1) = 10, K(-1,-1) = 17$, and $K(1,-1) = 8$;
ACI (two-stage): ACI model with two stage minimum D_K-distance estimator $\hat{\phi}^{opt}$, using the preliminary estimation result of **ACI (one-stage)** as the first stage.

5.1.1. Forecast criteria

We employ eleven criteria to evaluate forecast accuracy for crude oil prices. First, ω_1 and ω_2 directly evaluate the monthly mean forecasts of oil price intervals; they are defined as follows:

$$\omega_1 = 1 - \frac{1}{S}\sum_t 1\left(FA_t \in \left[\hat{L}_t, \hat{H}_t\right]\right), \tag{5.1}$$

$$\omega_2 = 1 - \frac{1}{S}\sum_t \frac{\min(\hat{H}_t, H_t) - \max(\hat{L}_t, L_t)}{\max(\hat{H}_t, H_t) - \min(\hat{L}_t, L_t)}, \tag{5.2}$$

where $[\hat{L}_t, \hat{H}_t]$ and $[L_t, H_t]$ are the forecasted and actual intervals of oil price in month t; $1(\cdot)$ denotes the indicator function with $1(\cdot) = 1$ if the monthly averaged price $FA_t \in [\hat{L}_t, \hat{H}_t]$ and $1(\cdot) = 0$ otherwise. The summations are taken over the forecast periods of $S = 183$ ($S = 123$). Intuitively, ω_1 measures the percentage of monthly average crude oil prices which do not fall into the forecasted intervals, while ω_2 measures the nonoverlapping area of the actual and forecasted intervals. A smaller ω_1 or ω_2 implies better interval forecasts.

Next, we follow the spirit of Sun et al. (2018) to employ other five interval-based criteria to measure the dissimilarity between an observed interval and its forecast. One is defined as

$$\omega_{MDE} = \frac{1}{S}\sum_t \sqrt{|\hat{M}_t - M_t|^2 + |\hat{R}_t - R_t|^2}, \tag{5.3}$$

which is a special case of the mean distance error (MDE) defined in Arroyo et al. (2011). Two other forecast criteria used in Rodrigues and Salish (2015) and Sun et al. (2018) are defined as:

$$\omega_{NSD1} = \frac{1}{S}\sum_t \frac{w([L_t, H_t] \cup [\hat{L}_t, \hat{H}_t]) - w([L_t, H_t] \cap [\hat{L}_t, \hat{H}_t])}{w([L_t, H_t] \cup [\hat{L}_t, \hat{H}_t])}, \tag{5.4}$$

$$\omega_{NSD2} = \frac{1}{S}\sum_t \frac{w([\hat{L}_t,\hat{H}_t]) + w([L_t,H_t])}{R([L_t,H_t]\cup[\hat{L}_t,\hat{H}_t])}, \tag{5.5}$$

where $w(\cdot)$ denotes the width of an interval, $R(\cdot)$ denotes the range of an interval, $\cap$ is the intersection, and $\cup$ is the union. These two criteria measure the normalized symmetric difference (NSD) of intervals, which can be considered as the nonoverlapping area of the actual and forecasting intervals. In addition, the noncoverage rate and nonefficiency rate in Rodrigues and Salish (2015) and Sun et al. (2018) are defined as

$$\omega_c = 1 - \frac{1}{S}\frac{w([L_t,H_t]\cap[\hat{L}_t,\hat{H}_t])}{w([L_t,H_t])}, \tag{5.6}$$

$$\omega_e = 1 - \frac{1}{S}\frac{w([L_t,H_t]\cap[\hat{L}_t,\hat{H}_t])}{w([\hat{L}_t,\hat{H}_t])}. \tag{5.7}$$

For each of these five criteria, a smaller value implies a better interval forecast.

Third, another four point-based criteria are the root mean square error (RMSE) in forecasting four-point attributes, i.e., the midpoint M_t, the range R_t, the low L_t and the high H_t, respectively. With similar notations in (5.1) and (5.2), we have:

$$\omega_M = \sqrt{\frac{1}{S}\sum_t [\hat{M}_t - M_t]^2}, \tag{5.8}$$

$$\omega_R = \sqrt{\frac{1}{S}\sum_t [\hat{R}_t - R_t]^2}, \tag{5.9}$$

$$\omega_L = \sqrt{\frac{1}{S}\sum_t [\hat{L}_t - L_t]^2}, \tag{5.10}$$

and

$$\omega_H = \sqrt{\frac{1}{S}\sum_t [\hat{H}_t - H_t]^2}. \tag{5.11}$$

Finally, the interval U of Theil statistics (U^I) used in Maia and de Carvalho (2011) is defined as

$$U^I = \sqrt{\frac{\sum_t (H_{t+1} - \hat{H}_{t+1})^2 + \sum_t (L_{t+1} - \hat{L}_{t+1})^2}{\sum_t (H_{t+1} - H_t)^2 + \sum_t (L_{t+1} - L_t)^2}}.$$

The U^I statistic is suitable for comparing forecast errors with the random walk benchmark. $U^I < 1$ implies that a forecasting model outperforms a random walk model, while $U^I \geq 1$ implies that the forecasting model underperforms a random walk.

5.1.2. Forecasting results analysis

Tables 3 and 4 present an assessment of quality of mean forecasting results of various models using the aforementioned criteria for interval forecasts. Results without and with the speculation variable are reported in Panels A and B, respectively. We have the following important findings:

First, the ACI model with the two-stage estimator $\hat{\phi}^{opt}$ generally delivers the best forecasting results in most cases. One possible explanation is that it treats an interval observation as a set and models the interval data directly, which more efficiently utilizes the rich information contained in an interval data and so improves forecast accuracy. Another reason is the use of an

Table 3. Interval and point forecast performance criteria during 2003–2018.

Panel A: Forecasting models without speculation						
Criteria	ARMAX	Bivariate MR	Bivariate LH	VEC	ACI (One stage)	ACI (Two stage ϕ^{opt})
ω_1	0.5191	0.5246	0.5410	0.4918	0.5246	0.5191
ω_2	0.6693	0.6776	0.7590	0.6472	0.6520	0.6518
ω_{MDE}	0.0885	0.0927	0.1050	0.0849	0.0841	0.0840
ω_{NSD1}	0.6601	0.6676	0.6542	0.6383	0.6453	0.6447
ω_{NSD2}	0.6693	0.6918	0.7590	0.6472	0.6520	0.6518
ω_c	0.5200	0.5956	1.0210	0.5001	0.5063	0.5052
ω_e	0.5747	0.6381	0.5311	0.5486	0.5524	0.5533
ω_M	0.1088	0.1042	0.1244	0.0874	0.0853	0.0858
ω_R	0.0602	0.0827	0.1101	0.0582	0.0581	0.0574
ω_L	0.1246	0.1207	0.1633	0.1043	0.1027	0.1023
ω_H	0.0999	0.1029	0.1016	0.0780	0.0755	0.0768
U^I	1.0356	1.1341	1.4720	0.9960	0.9750	0.9787
Panel B: Forecasting models with speculation						
Criteria	ARMAX	Bivariate MR	Bivariate LH	VEC	ACI (One stage)	ACI (Two stage ϕ^{opt})
ω_1	0.5355	0.5301	0.4973	0.4918	0.5191	0.4863
ω_2	0.6599	0.7461	0.6921	0.6500	0.6540	0.6471
ω_{MDE}	0.0870	0.1193	0.0884	0.0853	0.0857	0.0831
ω_{NSD1}	0.6501	0.6620	0.6465	0.6407	0.6467	0.6400
ω_{NSD2}	0.6599	0.7461	0.6921	0.6500	0.6540	0.6471
ω_c	0.5158	0.9592	0.5799	0.5070	0.5069	0.5000
ω_e	0.5648	0.4726	0.5425	0.5523	0.5585	0.5480
ω_M	0.0942	0.1531	0.0866	0.0881	0.0850	0.0842
ω_R	0.0593	0.1458	0.0828	0.0584	0.0616	0.0573
ω_L	0.1095	0.1508	0.1134	0.1050	0.1041	0.1007
ω_H	0.0867	0.1866	0.0746	0.0788	0.0742	0.0754
U^I	0.9975	1.8364	1.0383	1.0031	0.9778	0.9625

(i). ω_1 measures the percentage of monthly average crude oil prices which does not fall into the forecasted intervals; ω_{MDE} based on the mean distance error of intervals evaluates the forecasting interval as a whole; ω_2, ω_{NSD1} and ω_{NSD2} measure the nonoverlapping area of the actual and forecasting intervals; ω_c and ω_e measure the noncoverage rate and nonefficiency rate; $\omega_M, \omega_R, \omega_L$ and ω_H measure the RMSE of the midpoints, ranges, lows and highs, respectively; and U^I is the Theil statistics. The smaller values of these measures, the better performance of the forecasts.
(ii). The statistics are computed over $S = 183$ forecasting periods.

estimated optimal kernel, which not only downweights large residuals of the left and right bounds but also correct correlations between them.

Second, compared with the forecasting results of the ARMAX model with the CLS estimator using midpoint and range data separately, the ACI model with two-stage interval-based estimator $\hat{\phi}^{opt}$ has gain in terms of forecasting ranges and midpoints, yielding smaller ω_R and ω_M than the ARMAX models. It confirms that the level (midpoint) information contained in an interval data will help even when the interest is in forecasting range only.

Third, turning to the derived bivariate point-based ARMAX models with CCLS estimation, we find that when K_{MR} is used to assign equal weights to range and midpoint observations for the bivariate MR model, but ignore the possible correlation between them, the ACI model with the two-stage estimator $\hat{\phi}^{opt}$ outperforms the bivariate MR model in terms of almost all forecast criteria. Similar results are obtained when K_{LH} for the bivariate LH model is used, which assigns an equal weight to the left and right bound observations but ignores their correlations. Again, the ACI model with the two-stage estimator $\hat{\phi}^{opt}$ performs better, thereby indicating the valuable information contained in interval time series data.

Fourth, when the speculation term is added, the forecasting results of the interval-based and point-based attributes all support that the ACI model with the two-stage minimum D_K-distance estimator outperforms the VEC model.

Furthermore, results in Panels A and B imply that including the speculation term SPE_t generally improves upon the forecasting results for the models of ARMAX, bivariate LH and ACI

Table 4. Interval and point forecast performance criteria during 2008–2018.

Criteria	ARMAX	Bivariate MR	Bivariate LH	VEC	ACI (One stage)	ACI (Two stage ϕ^{opt})
Panel A: Forecasting models without speculation						
ω_1	0.5122	0.5122	0.5203	0.4634	0.5041	0.4959
ω_2	0.6753	0.6512	0.8062	0.6449	0.6476	0.6464
ω_{MDE}	0.0946	0.0899	0.1196	0.0902	0.0884	0.0882
ω_{NSD1}	0.6636	0.6397	0.6543	0.6342	0.6422	0.6400
ω_{NSD2}	0.6753	0.6512	0.8062	0.6449	0.6476	0.6464
ω_c	0.5256	0.5142	1.2609	0.4946	0.4957	0.4925
ω_e	0.5820	0.5884	0.5177	0.5497	0.5494	0.5501
ω_M	0.0978	0.0944	0.1430	0.0939	0.0906	0.0912
ω_R	0.0675	0.0632	0.1305	0.0645	0.0637	0.0628
ω_L	0.1182	0.1134	0.1909	0.1142	0.1113	0.1107
ω_H	0.0863	0.0834	0.1140	0.0818	0.0779	0.0796
U^I	1.0440	1.0046	1.5867	1.0022	0.9696	0.9732
Panel B: Forecasting models with speculation						
Criteria	ARMAX	Bivariate MR	Bivariate LH	VEC	ACI (One stage)	ACI (Two stage ϕ^{opt})
ω_1	0.5528	0.5203	0.4878	0.4715	0.5041	0.4634
ω_2	0.6762	0.7526	0.7068	0.6495	0.6514	0.6395
ω_{MDE}	0.0935	0.1331	0.0943	0.0906	0.0906	0.0868
ω_{NSD1}	0.6630	0.6594	0.6430	0.6378	0.6447	0.6330
ω_{NSD2}	0.6762	0.7526	0.7068	0.6495	0.6514	0.6395
ω_c	0.5409	1.1108	0.6066	0.5046	0.4981	0.4849
ω_e	0.5814	0.4045	0.5350	0.5556	0.5593	0.5419
ω_M	0.0940	0.1757	0.0930	0.0949	0.0904	0.0891
ω_R	0.0674	0.1697	0.0957	0.0646	0.0682	0.0627
ω_L	0.1144	0.1679	0.1263	0.1151	0.1134	0.1086
ω_H	0.0828	0.2189	0.0769	0.0827	0.0761	0.0777
U^I	1.0071	1.9692	1.0554	1.0116	0.9748	0.9528

(i). ω_1 measures the percentage of monthly average crude oil prices which does not fall into the forecasted intervals; ω_{MDE} based on the mean distance error of intervals evaluates the forecasting interval as a whole; ω_2, ω_{NSD1} and ω_{NSD2} measure the nonoverlapping area of the actual and forecasting intervals; ω_c and ω_e measure the noncoverage rate and nonefficiency rate; $\omega_M, \omega_R, \omega_L$ and ω_H measure the RMSE of the midpoints, ranges, lows and highs, respectively; and U^I is the Theil statistics. The smaller values of these measures, the better performance of the forecasts.
(ii). The statistics are computed over $S = 123$ forecasting periods.

(two-stage) that do not include SPE_t, suggesting the predictive power of speculation in forecasting crude oil prices. These results are robust to various forecast periods.

5.2. Forecasting the volatility of crude oil prices

In this section, we conduct out-of-sample forecasts and evaluate the performance of the ACI model in forecasting volatility of oil prices. Our benchmarks are the ARMA(2,2)-GJR(1,1)-skewed *t* and CARR models introduced in Section 4.2.

We use two measures of the *ex post* volatility: monthly return squared (MRSQ) and the sum of squared daily returns (SSDR). SSDR is calculated by aggregating the squared daily returns within each month. It is the realized volatility. The threshold GARCH model directly forecasts the conditional variances of return series with SSDR and MRSQ. For the CARR and ACI models, following the spirit of Parkinson (1980),[7] the conditional variance forecast $\hat{h}_t$ is obtained by the following transformation from the range forecast $\hat{R}_t$:

$$\hat{h}_t = \hat{R}_t^2/(4 \times \ln(2)).$$

[7]Suppose a point particle undergoes a one-dimensional continuous random walk with a diffusion constant *D*. Parkinson (1980) mentioned that the difference *l* between the maximum and minimum positions is a good estimator for the diffusion constant.

Table 5. Out-of-sample forecast of variance.

	SSDR			MRSQ		
	MSE (10^{-4})	MAD (10^{-4})	*p* Value	MSE (10^{-4})	MAD (10^{-4})	*p* Value
Panel A: Out-of-sample forecast during 2003–2018						
ACIX	1.5027	61.1661		1.6085	69.8612	
GARCHX	2.0225	65.5813	0.0072	1.7820	71.7679	0.0033
CARRX	1.6297	61.3659	0.0101	1.6695	68.5758	0.0023
ACI	1.5348	61.2317		1.6399	69.8548	
GARCH	1.9911	64.9084	0.0080	1.7981	72.5542	0.0034
CARR	1.6446	60.6901	0.0167	1.6567	67.7679	0.0026
Panel B: Out-of-sample forecast during 2008–2018						
ACIX	1.9983	67.5978		2.2055	80.5409	
GARCHX	2.7686	76.4896	0.0130	2.4639	83.0813	0.0061
CARRX	2.2094	69.7009	0.0187	2.3184	79.4855	0.0045
ACI	2.0409	67.4450		2.2506	80.7254	
GARCH	2.7567	77.0499	0.0132	2.4983	85.2018	0.0061
CARR	2.2402	69.6156	0.0280	2.3132	79.0811	0.0046

(i). This table computes the mean-absolute-errors (MAD) and the mean-squared-errors (MSE) of conditional variance forecasts of ACIX, GARCH, CARR models. *p* Value denotes the *p*-value of the Diebold-Mariano test for the significant outstanding performance recorded by the ACI and ACIX models, respectively.
(ii). Two measured volatility are used. SSDR and MRSQ are the sum of squared daily returns over the month and the monthly return squared, respectively.
(iii). GARCHX is fitted for the monthly return series with speculation, the CARRX is fitted for the monthly range series with speculation, and the ACIX is fitted for the monthly interval-valued data series with speculation.
(iv) The data used are from January 1993 to March 2018. Rolling samples of 120 observations are used in fitting the models and 183 and 123 observations are made for the out-of-sample forecasts, respectively.

Then, we compute the following mean-squared-erros (MSE) and the mean-absolute-errors (MAD):

$$MSE(MRSQ) = S^{-1}\sum_t (MRSQ - \hat{h}_t)^2, \tag{5.12}$$

$$MSE(SSDR) = S^{-1}\sum_t (SSDR - \hat{h}_t)^2, \tag{5.13}$$

$$MAD(MRSQ) = S^{-1}\sum_t |MRSQ - \hat{h}_t|, \tag{5.14}$$

$$MAD(SSDR) = S^{-1}\sum_t |SSDR - \hat{h}_t|. \tag{5.15}$$

Table 5 reports the results of the two forecast evaluation criteria and *p*-values of the Diebold-Mariano test. Models with (labeled with X in Table 5) and without the speculation variable are examined. First, under the both criteria, the ACI models show an overwhelming pattern in producing better volatility forecasts than the threshold GARCH models. The gain is remarkable when SSDR is used under MSE criterion, where the MSE of the ACIX model with speculation is about 25% smaller than that of the GARCHX model with speculation.

Second, in comparison with the CARR models, the ACI models have better forecasts for volatility in most cases. The differences in the performance between the two models are observed for both SSDR and MRSQ forecasts. As consistent with the threshold GARCH results, the improvement of forecasting accuracy by the ACI models is usually much more obvious when SSDR is used as a measure for realized volatility. This is a strong evidence for the advantage of ACI models, as the realized volatility, i.e., SSDR, uses more information (daily) and thereby is a more precise volatility measure than the MRSQ. Meanwhile, a closer examination of the results shows that the ACI models always outperform the CARR models when the evaluation criterion MSE is used. This indicates that the forecasting accuracy of ACI models are more stable than CARR models.

Furthermore, the p-value for the Diebold-Mariano test is almost smaller than 10%, which provides significant evidence for the superior performance of the ACI models to other existing models at the 10% significance level. All these results are robust to various forecast periods.

5.3. Trading strategy based on monthly interval forecasts: VEC vs ACI

As indicated in Section 5.1, the ACI model with the two-stage minimum D_K-distance estimator generally performs the best in out-of-sample mean forecasts within the ones derived from special kernels, and it outperforms the VEC model in the majority of scenarios. This section further investigates the relative performance between the VEC model and the ACI model with the two stage estimator under a crude oil trading strategy in spirit of He et al. (2010). This trading strategy makes use of monthly low-high forecasts of the oil price intervals in Section 5.1.

Let $\hat{L}_{t+1}$ and $\hat{H}_{t+1}$ be the one-step-ahead low and high forecasts of the crude oil price in month t, produced after the t-th trading month is closed, and O_t and C_t denote the opening and closing prices of the crude oil price in month t respectively. The monthly opening price O_t is obtained by the average of the daily opening prices in month t. The trading rule is designed as follows: A buy alert signal is generated if $\hat{H}_{t+1} - O_t > O_t - \hat{L}_{t+1}$, after the t-th trading month is closed. If the buy alert signal maintains for m consecutive months including month t, we place a buy order in month $t+m-1$ using the closing price of C_{t-m+1}. Assuming that no short sale is allowed, then a sell alert signal is generated in month s $(s > t)$ if $\hat{H}_{s+1} - O_s < O_s - \hat{L}_{s+1}$. A sell will be made if the sell alert signal is observed for another m consecutive months including month s with the execution price of C_{s-m+1}. The investor can act according to the buy alert even if she or he holds the previous positions. It is possible that there are unsold positions at the end, thereby not being taken into account in profit calculation, because the sell alert signal has appeared for less than m months in the end of the trading period.

The evaluation criterion of this trading strategy is based on the annualized returns (AR), namely,

$$AR = \left(\frac{C_{t+j} - C_t}{C_t} \times 100\% - 0.1\% \right) \times \frac{12}{j} \tag{5.16}$$

where C_{t+j} and C_t are the monthly closing prices of the selling and buying months, respectively. When the profit of each trade is calculated, a one-way 0.1% deduction is considered to mimic transaction cost.[8] For details about this trading rule on daily basis, readers are referred to He et al. (2010).

Before comparing the ACI models and the point-valued time series models, a simple buy-and-hold trading rule is considered as an initial benchmark. An investor buys an asset at the beginning and sells it at the end of the evaluation period. The annualized return of this naive trading rule is 7.35%, which is smaller than the best of averaged annualized returns 10.01% for the VEC model and 18.86% for the ACI model. Obviously, the trading rule based on highs and lows generates substantial economic gains.

The AR values obtained from the trading rule based on highs and lows are shown in Table 6, where the results of the VEC and ACI models are reported on the left part and right part of the table, respectively. Several patterns are immediately observed. First, using interval forecasts from the ACI model generates more profits than the VEC model under most scenarios, in terms of the

[8]In the futures market, there is a practice of daily settlement or marking to market. Investors may take a risk arising from large margin calls. Therefore, we also compute daily gains or losses of long positions before every sell action is triggered during the evaluation period. It is found that the trading obtained from an ACI model usually has a smaller margin account risk than that from the point-valued time series model for $m = 2$, 3, and 4, respectively. This indicates again that the use of interval information can improve profitability of crude oil modeling. The relevant empirical results are available from the authors upon request.

Table 6. Performance of VEC and ACI models of trading strategy.

	VEC		ACI	
	With speculation	No speculation	With speculation	No speculation
Panel A: $m=1$				
Averaged AR	−6.288	−8.676%	−11.430%	−12.079%
Largest AR from any trade	124.460%	124.460%	98.384%	80.048%
Smallest AR from any trade	−313.922%	−313.922%	−179.706%	−179.706%
Number of trades	127	127	128	119
% of trades with positive AR	56.693%	55.906%	50.000%	47.899%
Sharpe ratio	−0.110	−0.140	−0.239	−0.241
Panel B: $m=2$				
Averaged AR	10.014%	−24.593%	−5.293%	−5.124%
Largest AR from any trade	125.914%	52.655%	66.649%	69.266%
Smallest AR from any trade	−268.449%	−268.449%	−170.481%	−170.481%
Number of trades	93	92	104	94
% of trades with positive AR	66.667%	34.783%	51.923%	56.383%
Sharpe ratio	0.117	−0.477	−0.143	−0.143
Panel C: $m=3$				
Averaged AR	−19.036%	−21.510%	18.857%	1.954%
Largest AR from any trade	18.982%	15.261%	102.328%	66.649%
Smallest AR from any trade	−238.334%	−238.334%	−115.032%	−96.405%
Number of trades	69	67	76	66
% of trades with positive AR	30.435%	35.821%	63.158%	60.606%
Sharpe ratio	−0.496	−0.502	0.345	0.003
Panel D: $m=4$				
Averaged AR	−13.004%	−16.613%	15.092%	16.626%
Largest AR from any trade	8.633%	8.633%	94.290%	94.290%
Smallest AR from any trade	−179.097%	−179.097%	−94.437%	−94.437%
Number of trades	47	44	59	50
% of trades with positive AR	31.915%	31.82%	49.153%	50.000%
Sharpe ratio	−0.514	−0.503	0.271	0.303

(i). All return figures in Table 6 are expressed in annualized terms: $AR = (\frac{C_{t+j}-C_t}{C_t} \times 100\% - 0.1\%) \times \frac{12}{j}$, where C_{t+j} and C_t are the monthly closing prices of the selling and buying months respectively, 0.1% is the transaction cost of each trade.
(ii). The forecast horizon m carries over 1,2,3, and 4 months, and the corresponding results are reported in Panels A–D, respectively.

averaged AR and the percentage of trades with positive profits. Though ACI forecasts yield slightly less profits than VEC forecasts when $m=1$, it has a remarkable gain over the VEC model under the other cases. For example, when $m=4$ and the ACI model with speculation is used, the averaged AR is 15.09% and the frequency of profitable trades is about 49.15%. However, the VEC model with speculation delivers an averaged percentage loss of −13% and the frequency of positive profits is only 31.92%.

Second, the averaged AR generally increases for ACI models with increasing m, which can be viewed as a measure for the persistence of oil price momentum. In particular, the most profitable result under this trading strategy appears for the ACI model with speculation and $m=3$, where the averaged AR reaches 18.86%, and over 63.16% of the total trades generate positive returns. On contrary, when VEC forecasts are used, the trading rule results in a negative averaged percentage profit and more negative returns than positive returns for most cases, with exception on the VEC model with speculation and $m=2$.[9]

[9]When the trading rule for $m=2$ is considered, the VEC model with and without speculation deliver different trading results during the period from 2004 to 2008. As it is well known, crude oil prices had increased markedly in those years. In May 2008, WTI crude oil futures prices reached as high as U.S. \$125.46 per barrel. As a result, a number of slightly different buy-sell actions can lead to a great difference in the trading profits. This explains why the two types of VEC models differ greatly in the average returns for $m=2$.

When assessing the robustness of the trading performance, there is an overwhelming pattern on the largest and smallest ARs between the VEC and ACI models. Specifically, the extreme loss of the ACI model is much smaller than the VEC model across all cases, together with a higher extreme percentage profit in most cases. For example, when $m = 4$ the worst percentage lost of ACI forecasts is only half of that when VEC forecasts are used, while the magnitude of the best annualized profit of ACI forecasts can be as high as 11 times of that of the VEC forecasts.

We also employ the Sharpe ratio to evaluate the performance of various trading results:

$$SR = \frac{\mu}{\sigma}, \tag{5.17}$$

where μ and σ are the sample mean and the sample standard deviation of excess ARs, respectively. From Table 6, we can draw the same conclusion that the ACI model outperforms the VEC model in most cases. The trading strategies based on ACI forecasts yield higher Sharpe ratios than most of the investments based on VEC forecasts. In particular, for $m = 3$ and $m = 4$, the Sharpe ratios based on ACI models are all positive, while those based on VEC forecasts are negative. The largest value of Sharpe ratio based on ACI forecasts is 0.345, which is about three times of that based on VEC forecasts.

Note that the ACI model is much more parsimonious than the VEC counterpart. This implies that the latter may capture more subtle features of oil prices in the short run, and hence yields a little bit greater averaged ARs (although negative) in most cases when an investor uses forecasts for only one or two months. However, a complex model usually tends to over-fit price dynamics. As a result, the ACI model-based strategies have greater averaged ARs for $m = 3$, 4. Overall speaking, the ACI methodology can produce stable and profitable results when the low-high based trading strategy is adopted.

6. Conclusion

So far, most studies in crude oil price forecasts in the literature focus on point-valued closing price data and few have made good use of the interval-valued price data which contains rich information about the price dynamics. An interval can be characterized by two pairs of attributes: the lower (low) and upper (high) bounds, or equivalently the midpoint and range at a discrete time interval (e.g., daily, monthly), which are key components of trading in commodity and financial markets. The goal of this article is to provide a unified and parsimonious framework to forecast interval-valued crude oil prices. We use the newly proposed ACI models to forecast monthly crude oil prices, which are tailored to some important features of the crude oil market and are estimated by the minimum D_K-distance method. Rich information contained in interval-valued oil price observations can be simultaneously utilized, thus enhancing parameter estimation efficiency and model forecasting accuracy.

The data used in this paper are monthly interval-valued WTI crude oil futures prices from January 1993 to March 2018. We have the following important empirical findings:

- Compared with existing point-based models, the modified ACI models proposed by this paper improve the out-of-sample forecasting of monthly oil price intervals, as well as the forecasts for highs, lows, and ranges of oil prices.
- For volatility forecasts, the ACI models also beat the popular return-based threshold GARCH and range-based CARR models. Compared with the range CARR model, the ACI models generate superior forecasts for realized volatility and monthly squared returns in terms of the MSE criterion.
- The interval-valued speculation improves the forecast accuracy of crude oil prices and volatility in ACI models, while this improvement disappears in other point-based methods. This

implies that our interval modeling framework has an informational advantage of utilizing rich information contained in interval data over point-valued models.

- In order to illustrate an economic benefit of high and low forecasts, a trading strategy is examined. Substantially more profits are usually obtained under the interval-based ACI forecasts than the point-based VEC forecats.

In sum, the superior out-of-sample forecast performance of the ACI models confirms that the ACI models with the minimum D_K distance estimation can efficiently utilize the rich information in interval-valued time series data, which can be explored to improve estimation efficiency and forecast accuracy. Our empirical findings have potentially important implications for policy makings and decision makings in crude oil markets.

It would be interesting to extend the modified ACI models to more sophisticated interval models to capture nonlinear features of an interval time series process, which may be more effective for forecasting volatility, highs, and lows of an interval time series. On the other hand, it will be also desirable to compare the out-of-sample volatility forecasts of the modified ACI models with other competing volatility models in the existing literature (Ewing et al., 2019; Hansen and Lunde, 2005; Herrera et al., 2018; Mei et al., 2019; Zhang et al., 2019). Furthermore, one can also apply the ACI models and their generalizations to investigate various aspects of risk management such as optimal hedging ratio, economic equivalence, and value-at-risk. All these topics are left for future work.

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